Analysis of the Low-Energy Theorem for $\gamma p \to p\pi^0$

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The derivation of the 'classical' low-energy theorem (LET) for $\gamma p \to p\pi^0$ is re-examined and compared to chiral perturbation theory. Both results are correct and are not contradictory; they differ because different expansions of the same quantity are involved. Possible modifications of the extended partially conserved axial-vector current relation, one of the starting points in the derivation of the LET, are discussed. An alternate, more transparent form of the LET is presented.

The low-energy theorem (LET) for photoproduction of neutral pions from protons is the subject of an on-going discussion. The reason for this is that the recent calculations in the framework of chiral perturbation theory (CHPT) [1,2] and heavy baryon chiral perturbation theory (HBCHPT) [3] are claimed to contradict the older result, i.e. the 'classical' LET [4]. This lead to the statement that the LET [5] is actually not a theorem [1–3,6]. In this Letter, we point out that the LET is a theorem in the sense that it is based on a few general principles, and once these are given the final result is model-independent. It is also shown that the commonly used extrapolation [4,7] of the off-shell to the physical pion-nucleon coupling can be avoided by means of the exact, i.e. non-extrapolated Goldberger-Treiman relation. Furthermore, the validity of the particular form of chiral symmetry breaking used in deriving the LET is investigated. However, as will be discussed below, the apparent discrepancy between the CHPT [8] calculation to order q^3 and the LET is due to the fact that expansions in different parameters, pion mass versus energy, are made. Therefore, both results are correct within their respective frameworks.

A model-independent result based on a few general principles should be regarded as a theorem. The LET under consideration is based on Lorentz invariance, gauge invariance, crossing symmetry, the partially conserved axial-vector current (PCAC) *hypothesis*, and its extension to include the electromagnetic interaction [9]. This hypothesis formulates the underlying chiral symmetry and its breaking. Note that, besides spontaneous, also explicit symmetry breaking is included. Recently, the LET has been carefully rederived [7] starting from the principles mentioned above. The consequences of isospin symmetry breaking [10] and the explicitly broken chiral symmetry for the LET [11] have also been addressed. The LET was shown not to be modified. Let us recall some details relevant for this work, which mainly concern the implementation of extended PCAC,

$$(i\partial^\mu + e_\pi A^\mu) J^{\pm,0}_{5,\mu} = i f_\pi M_\pi^2 \phi^{\pm,0} \, . \tag{1}$$

Here, A_μ is the photon field, $J_{5,\mu}$ is the axial-vector current, ϕ is the pion (interpolating) field, M_π is the pion mass, f_π is the pion decay constant, and e_π is the pion charge. Taking the appropriate matrix element of Eq. (1), one finds

$$\frac{f_\pi M_\pi^2}{M_\pi^2 - q^2} \langle N(\vec{p}\,')|j_\pi^{\pm,0}|N(\vec{p}), \gamma(\vec{k})\rangle = i q^\mu \langle N(\vec{p}\,')|J^{\pm,0}_{5,\mu}|N(\vec{p}), \gamma(\vec{k})\rangle - e_\pi \langle N(\vec{p}\,')|J^{\pm,0}_{5,\mu} A^\mu|N(\vec{p}), \gamma(\vec{k})\rangle \, . \tag{2}$$

The left-hand side of this equation contains the pion production amplitude expressed by means of the pion source, j_π. The second term on the right-hand side of Eq. (2) does not contribute to neutral pion production. The first term on the right-hand side of Eq. (2) requires a careful treatment because of possible nucleon pole contributions. Therefore, the amplitude is first divided into two general classes: class A diagrams, which contain the dressed nucleon propagator and half off-shell γNN and πNN vertices, and generalized non-pole contributions (class B diagrams). As the above mentioned principles provide enough constraints on these contributions to determine the LET, no (microscopic) model or theory for the hadron structure is needed. This does not imply, however, that the internal structure of the hadrons is ignored.

Since Eq. (2) is a priori defined only for virtual pions [7], the production of off-shell pions with $\vec{q} = 0$ (the pion three momentum in the cm frame) and $q_0 \to 0$ (the pion energy in the cm frame), has been considered. This does not mean that the pion mass is taken to be zero. As a technical tool, an artificial mass difference between the in- and outgoing nucleons is introduced [4] (this procedure is not unique and other methods, yielding the same final result, have been used [12]). In this way, two small parameters appear, $\omega = q_0/M$ [13] and $\delta = (M' - M)/M$. These mathematical tools are necessary to determine unknown amplitudes. In particular, the mass splitting [14] enables one to separate out possible pole terms, i.e. contributions of order $1/q_0$. After this separation, it is assumed that an expansion of the amplitude in ω and δ is valid, with δ taking on its physical value in the end. The expansion in ω is supposed to hold for $q_0 < M_\pi$ (the validity of this assumption and the limit $q_0 \to M_\pi$ will be discussed below).

Applying the above principles then leads to the final result, the LET for $\gamma p \to p\pi^0$

$$E_{0+} = -\frac{eg}{8\pi M}\left[\omega - \frac{\omega^2}{2}(3+\kappa_p)\right] + \mathcal{O}(\omega^3) \; , \tag{3}$$

where E_{0+} is the s-wave multipole. It is evidently an expansion in the kinematical variable ω, and only contains observable hadron properties like the nucleon mass, M, the proton charge, e, the proton anomalous magnetic moment, κ_p, and the pion-nucleon coupling constant, g. In order to compare with experiment, the limit $\omega \to M_\pi/M \equiv \mu$ is normally taken in Eq. (3), and the LET is usually presented at the point $\omega = \mu$. However, to exploit the main power of the LET, i.e. a theoretical check on models, the above limit is not required [15].

In CHPT, one makes the plausible assumption [16] that the low-energy regime of QCD is described by an effective Lagrangian [17] incorporating the global symmetries of QCD. One then arrives at a systematic expansion scheme in terms of small momenta and meson masses, denoted by q^n. Up to order q^3, the CHPT result [1–3] is

$$E_{0+} = -\frac{eg}{8\pi M}\left[\mu - \frac{\mu^2}{2}(3+\kappa_p + \frac{M^2}{8f_\pi^2})\right] + \mathcal{O}(\mu^3) \; . \tag{4}$$

It should be remarked that Eq. (4) is valid at threshold, and that no distinction is made between the pion energy and the pion mass. In other words, the CHPT amplitude is expanded in the pion mass regardless of its origin. It is obvious that Eqs. (3) and (4) disagree at $\omega = \mu$; numerically, Eq. (3) gives -2.3 $\times$ $10^{-3}M_{\pi^+}^{-1}$, while Eq. (4) gives +0.9 in the same units. The current experimental result [12,18,19] is -2.0 $\pm$ 0.2, and new experiments are underway at SAL and Mainz.

On the theoretical side, we do not object to the conclusive remark in [6], but we propose to extend it to include the LET: for models that satisfy extended PCAC, CHPT *and* the LET *simultaneously* serve as important theoretical checks. Nevertheless, we do not agree with the criticisms in Ref. [6], which state that several assumptions were used in the derivation of the LET which do not hold in the 'standard model'. The reason is that these assumptions were not made in the derivation of the LET, and we now successively address them. First, we stress that all loop contributions are implicitly taken into account in the derivation of the LET. The general vertices and propagators, as well as the general non-pole contributions, in principle contain loops, and no contribution is dropped or neglected.

Secondly, it is often stated [1–3,6] that one assumes that the coefficients of the ω expansion are analytic in μ, and as this is not true in CHPT, the LET is wrong. The divergence of some coefficients in the chiral limit is nothing else than the Li-Pagels mechanism [20]. Already before the CHPT calculations, this potential problem was addressed in detail, and it was shown that the LET does not change due to this [11]. It was also anticipated that the coefficients of the μ expansion could be different than the coefficients of the ω expansion due to the Li-Pagels mechanism. Differentiating between pion mass and energy, and only expanding in the latter, avoids this mechanism and the LET is not affected. Thus, the assumption stated above has not been made, and the LET is valid even if the coefficients are nonanalytic in μ. We emphasize once more that an expansion has been made in the variables ω and δ. After implementation of PCAC, the fictitious mass difference, δ can be put to zero and one is left with an expansion in the energy, ω. Usually, however, the on-shell limit, $\omega = \mu$, is taken, and the result takes on the *form* of an expansion in the pion-nucleon mass ratio. As already recognized by Kroll and Ruderman [21] and later by Vainsthein and Zakharov [22], there is no a priori reason that this expression should coincide with the μ expansion of the amplitude. The coefficients may also depend on μ and this dependence is not constrained by the principles used in the derivation of the LET. We stress that the validity of the ω expansion for $\omega < \mu$ is not disproved in the CHPT calculation. In fact, it was argued [23] that the expansion converges for $0 < \omega \leq \mu$. Morever, it was shown [23] that expansion of the CHPT amplitude [2] in terms of ω produces the LET, Eq. (3). In summary, QCD does not forbid an energy expansion with finite pion mass, and, as in Compton scattering, the coefficients of this expansion may be nonanalytic in the pion mass. We cannot, however, prove PCAC and its extension starting from QCD.

The arguments raised above can be explicitly checked in the linear sigma model [15] or from the CHPT amplitude [23]. In particular, one can test if the assumed power expansion in ω is valid for $\omega \leq \mu$. It was found for both the CHPT amplitude [23] and the linear sigma model amplitude [15] that the expansion converges for $\omega \leq \mu$, and the coefficients of ω^n (for $n \geq 3$) are $\sim 1/\mu^{(n-2)}$, in agreement with the anticipation of Li and Pagels [20]. It should be emphasized that to obtain the LET, only an expansion in ω is needed. In contrast to the claim in [6], no additional expansion in μ is needed. In order to illustrate the behavior of the ω expansion at $\omega = \mu$, the ω coefficients were expanded in μ [15], and it was demonstrated that an infinite series had to be summed to obtain the CHPT result. However, as the series converges at $\omega = \mu$, this does not imply that an 'illicit interchange of limits' [6] has been made. The off-shell behavior of the CHPT amplitude could, of course, be different than the off-shell behavior of the linear sigma model amplitude. In an effective Lagrangian approach, such as CHPT, off-shell ambiguities may arise due to the presence of terms which vanish on-shell, see, e.g. [24]. We re-emphasize that starting with the above principles,

the LET is valid in the region $\omega \leq \mu$ and it has no off-shell ambiguity, i.e. models that satisfy PCAC should agree on the off-shell value of the E_{0+} up to and including order ω^2.

Turning to the data, the experiment happens to agree with the numerical value of the LET, and, consequently, not with the order q^3 CHPT result. The disagreement between experiment and the CHPT result is believed to be due to the slow convergence of the expansion in μ, and it is concluded that this reaction is not the ideal place to test the standard model [6]. Although convergence issues are beyond the predictive power of LET's in general, one can look at convergence questions given a reasonable model. Indeed, the ω expansion of the linear sigma model amplitude [15] converges slowly as $\omega \to \mu$, and therefore the agreement of the LET with the data is somewhat surprising. Another problem in CHPT is that the isospin violation corrections are not fully understood, but expected to be large. In contrast, the LET for this reaction is only trivially modified [10] by isospin violation corrections. In the linear sigma model, the main source of difference between the exact model result and the LET value arises from the intermediate $n\pi^+$ state. In nature, this threshold is 6 MeV above the $p\pi^0$ threshold. Including isospin symmetry breaking (by hand) in the linear sigma model, it was found that the LET value is more accurate, but still 35% higher than the exact model result [15].

The result of the derivation in Ref. [7] actually contains the off-shell pion-nucleon coupling at $q^2 = 0$, i.e. $g(0)$. Of course, given the order of ω, it is consistent to replace it by $g(q_0^2)$. Finally putting in $\omega = \mu$ yields the LET in terms of the physical pion-nucleon constant, g. This procedure appears like the extrapolation usually made in the derivation of the Goldberger-Treiman relation. In fact, by use of the exact Goldberger-Treiman relation, i.e. *not* using the extrapolation, one can immediately express the result in terms of physical quantities. To demonstrate this explicitly, recall that PCAC yields a relation between $g(0)$ and the physical weak interaction constants g_A, the nucleon axial-vector coupling constant, and f_π,

$$g_A = \cos\theta_c \frac{f_\pi g(0)}{M} \ , \tag{5}$$

where θ_c is the Cabibbo angle. Given the PCAC hypothesis (one of the basic ingredients in the derivation) this is exact. What is commonly known as the Goldberger-Treiman relation follows from the extrapolation mentioned above, i.e., $g(0) \approx g(M_\pi^2)$. However, the LET involves $g(0)$ [7], and thus we can use the exact relation, Eq. (5). The final expression then reads

$$E_{0^+} = -\frac{e g_A}{8\pi f_\pi \cos\theta_c}\left[\omega - \frac{\omega^2}{2}(3+\kappa_p)\right] + \mathcal{O}(\omega^3) \ . \tag{6}$$

In the discussions above, we have refuted the criticisms of the LET given in Refs. [1–3,6] and re-established the LET based on extended PCAC. We note here that the extension of the PCAC relation was derived assuming minimal electromagnetic coupling [9]. In effective theories, however, non-minimal terms can be present. The pertinent question is whether these terms would change the extended PCAC relation. Consequently, the low-energy expansion could be modified by a contribution of the form $q_0 M_\pi^2$, for instance. As an explicit example we consider CHPT. Although such a contribution is not present in the order q^3 calculation of [2], it does appear at order q^4. Explicitly, one has, in the notation of [25], the non-minimal term [26]

$$\mathcal{L} = ia\, Tr\left[\bar{B}\gamma_5\sigma_{\mu\nu}(F^{+\mu\nu}\rho + \rho F^{+\mu\nu})B\right] \ , \tag{7}$$

where a is an unknown constant. Looking at the neutral pion-nucleon sector, we find

$$\mathcal{L} \sim eaM_u F^{\mu\nu}\left[2\bar{p}\gamma_5\sigma_{\mu\nu}p\pi^0 + \bar{n}\gamma_5\sigma_{\mu\nu}n\pi^0\right] \ . \tag{8}$$

The contribution to the E_{0^+} at the tree level is $\sim q_0 M_u \sim q_0 M_\pi^2$. As anticipated, this term, Eq. (7), also gives a contribution to the divergence of the axial-vector current,

$$\Delta\partial_\mu J^\mu_{5,0} \sim eaM_u F^{\mu\nu}\left[2\bar{p}\gamma_5\sigma_{\mu\nu}p + \bar{n}\gamma_5\sigma_{\mu\nu}n\right] \ . \tag{9}$$

Therefore, Eqs. (1) and (2) are modified in CHPT, obviously leading to the possibility of contributions $\sim \omega\mu^2$ in Eq. (3) [27]. This contribution is evidently supressed compared to the linear term in Eq. (3).

It is important to recall that in theories with minimal electromagnetic coupling the extension of PCAC(-like) relations is known [9]. For example, in QCD, the divergence of the axial-vector current is expressed in terms of quark fields. The resulting expression is an acceptable pion interpolating field [28], $\phi^{\pm,0} \sim \bar{q}\gamma_5\tau^{\pm 0}q$ [29]. This choice of the pion interpolating field immediately yields PCAC; since this cannot be proved, PCAC remains a hypothesis [28]. Electromagnetic coupling to the elementary quarks leads to the extended PCAC relation. In other words, given the

PCAC hypothesis starting from QCD, corresponding to the above choice of the interpolating pion field, its extension holds. Note, however, that the choice of the pion interpolating field in terms of quarks fields is not unique [29]. Other choices modify the PCAC relation and it would be interesting to study the possibilities and consequences of such modifications. In CHPT, for instance, PCAC is most likely modified at some order.

Probably the best known example of a LET is Compton scattering [30,31]. From the theoretical point of view, there is no discussion about this LET. On the other hand, before the connection with experiment can be made, a careful treatment of infrared divergences is needed [32]. In one of the two classical derivations, a discussion on infrared divergences is included. Low [30] explicitly states that the proof applies to all orders in the electromagnetic coupling, provided the virtual photons are given a fictitious mass λ. He cannot guarantee its validity in the limit $\lambda \to 0$. Surprisingly, the situation for neutral pion photoproduction is under better control. First, infrared divergences concerning virtual photons are not present because the LET is only valid to first order in the electromagnetic coupling. Secondly, there is no need to give the virtual pions a fictitious mass because they are already massive. A few clarifying remarks are in order. The fictitious mass difference for the nucleons was introduced to deal with the nucleon pole in the matrix element of the axial-vector current. This is not connected with infrared problems; moreover, in the end we can take the equal mass limit. Finally, the energy expansion holds from zero up to the pion threshold. Beyond this threshold one cannot make definite statements.

In summary, the 'classical' LET derived *assuming* the extended PCAC relation has been verified. In particular, the LET does not break down due to neglect of loops, as they are implicitly taken into account, nor due to assumptions about the expansion in the pion energy. It has been proposed to present the LET in the form of an energy expansion, reflecting its real content. Moreover, it was demonstrated that the extrapolation of the pion-nucleon coupling can be avoided by using the exact Goldberger-Treiman relation. Furthermore, we pointed out that non-minimal contact couplings, often appearing in effective theories, may effect the extended PCAC relation. This was explicitly shown in CHPT, and the consequences for its low-energy expansion were exhibited. We also commented on the PCAC hypothesis in the context of QCD, where, given PCAC, its extension follows. Finally, a brief discussion of issues related to the expansion of the Compton amplitude and its similarities to the expansion used in deriving the LET was presented.

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